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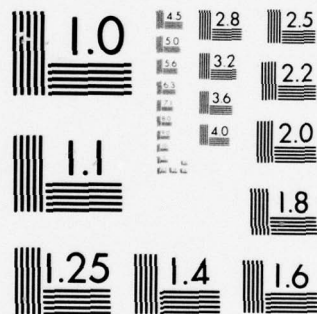
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A NOTE ON GENERATING GENERALIZED TWO-DIMENSIONAL  
PLATE AND SHELL THEORIES.

by E. Reissner

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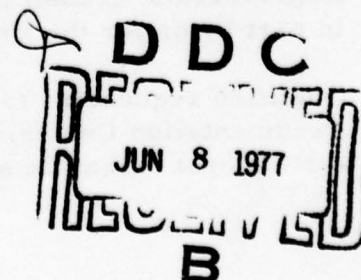
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A NOTE ON GENERATING GENERALIZED TWO-DIMENSIONAL  
PLATE AND SHELL THEORIES<sup>†</sup>

by

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ABSTRACT

"Ordinary" two dimensional linear shell theory may be formulated by using six differential equations for stress resultants and couples and for "middle" surface force and moment load intensities, in conjunction with the principle of virtual work, for the derivation of strain displacement relations. The present paper deals with a more general formulation, involving additional two-dimensional equilibrium equations, as a consequence of three-dimensional equations for force and moment stresses, in conjunction with a stipulation of surface force and moment load intensities for two face surfaces in place of the one middle surface. The main intent of the analysis is an illumination of the concept of a mechanical Cosserat-surface theory, in comparison with ordinary two-dimensional shell theory.

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<sup>†</sup> A report on work supported by the Office Of Naval Research, Washington, D. C.



# A NOTE ON GENERATING GENERALIZED TWO-DIMENSIONAL PLATE AND SHELL THEORIES

by

E. Reissner

## Introduction

Given a two-dimensional system of six scalar equilibrium differential equations for stress resultants and couples acting over the cross sections of an element of a shell, the associated strain displacement relations may be derived by an application of a suitable version of the principle of virtual work, with this derivation being of a particularly simple nature for the linear-theory problem [3]. In the following we wish to show briefly the nature of the result which is obtained upon deriving a more general system of two-dimensional equilibrium equations, as a consequence of a system of three-dimensional equilibrium differential equations for force and moment stresses, in conjunction with an appropriate stipulation of surface loads for the two faces of the layer in which the three-dimensional differential equations apply, again with a subsequent derivation of strain displacement relations through the principle of virtual work. The main purpose of our analysis is to throw additional light on the meaning of the concept of a Cosserat-surface two-dimensional shell theory, in comparison with "ordinary" two dimensional shell theory [1].

## The Three-Dimensional Boundary Value Problem

We assume a three-dimensional orthogonal coordinate system  $\xi_1, \xi_2, \zeta$ , where the  $\xi_1$  and  $\xi_2$ -curves are lines of curvature on the midsurface  $\zeta = 0$ , and where the  $\zeta$ -curves are straight lines perpendicular to the surface  $\zeta = 0$ , with the linear element being of the form  $[(1 + \zeta/R_1)\alpha_1 d\xi_1]^2 + [(1 + \zeta/R_2)\alpha_2 d\xi_2]^2 + d\zeta^2$ .

We introduce force and moment stress vectors  $\sigma_i^*, \sigma_\zeta^*, \tau_i^*, \tau_\zeta^*$ , as well as force and moment pseudo stress vectors  $\tilde{\sigma}_i, \tilde{\sigma}_\zeta, \tilde{\tau}_i, \tilde{\tau}_\zeta$ , given by

$$\begin{bmatrix} \tilde{\sigma}_1 \\ \tilde{\tau}_1 \end{bmatrix} = \left(1 + \frac{\zeta}{R_2}\right) \begin{bmatrix} \sigma_1^* \\ \tau_1^* \end{bmatrix}, \quad \begin{bmatrix} \tilde{\sigma}_2 \\ \tilde{\tau}_2 \end{bmatrix} = \left(1 + \frac{\zeta}{R_1}\right) \begin{bmatrix} \sigma_2^* \\ \tau_2^* \end{bmatrix}, \quad (1a)$$

$$\begin{bmatrix} \tilde{\sigma}_\zeta \\ \tilde{\tau}_\zeta \end{bmatrix} = \left(1 + \frac{\zeta}{R_1}\right) \left(1 + \frac{\zeta}{R_2}\right) \begin{bmatrix} \sigma_\zeta^* \\ \tau_\zeta^* \end{bmatrix}, \quad (1b)$$

with components representations

$$(\tilde{\sigma}_i, \tilde{\sigma}_\zeta) = (\sigma_{ij}, \sigma_{\zeta j}) \tilde{t}_j + (\sigma_{i\zeta}, \sigma_{\zeta\zeta}) \tilde{n}, \quad (2a)$$

$$(\tilde{\tau}_i, \tilde{\tau}_\zeta) = (\tau_{ij}, \tau_{\zeta j}) \tilde{n} \times \tilde{t}_j + (\tau_{i\zeta}, \tau_{\zeta\zeta}) \tilde{n}, \quad (2b)$$

where the  $\tilde{t}_j$  and  $\tilde{n}$  are tangent unit vectors to the coordinate curves. The eighteen distinct components of pseudo stress in (2a) and (2b) are readily shown to be subject to six differential equations of force and moment equilibrium which, in the here assumed absence of body force and moment loads, are of the form [4]

$$\frac{(\alpha_2 \sigma_{11})_{,1} + (\alpha_1 \sigma_{21})_{,2} + \alpha_{1,2} \sigma_{12} - \alpha_{2,1} \sigma_{22}}{\alpha_1 \alpha_2} + \frac{\sigma_{1\zeta}}{R_1} + \sigma_{\zeta 1, \zeta} = 0, \quad (3a)$$

$$\frac{(\alpha_2 \sigma_{12})_{,1} + (\alpha_1 \sigma_{22})_{,2} + \alpha_{2,1} \sigma_{21} - \alpha_{1,2} \sigma_{11}}{\alpha_1 \alpha_2} + \frac{\sigma_{2\zeta}}{R_2} + \sigma_{\zeta 2, \zeta} = 0, \quad (3b)$$

$$\frac{(\alpha_2 \sigma_{1\zeta})_{,1} + (\alpha_1 \sigma_{2\zeta})_{,2}}{\alpha_1 \alpha_2} - \frac{\sigma_{11}}{R_1} - \frac{\sigma_{22}}{R_2} + \sigma_{\zeta\zeta, \zeta} = 0, \quad (3c)$$

$$\frac{(\alpha_2 \tau_{11})_{,1} + (\alpha_1 \tau_{21})_{,2} + \alpha_{1,2} \tau_{12} - \alpha_{2,1} \tau_{22}}{\alpha_1 \alpha_2} + \frac{\tau_2 \zeta}{R_2} + \tau_{\zeta 1, \zeta} + \sigma_{\zeta 1} - \left(1 + \frac{\zeta}{R_1}\right) \sigma_{1 \zeta} = 0, \quad (3d)$$

$$\frac{(\alpha_2 \tau_{12})_{,1} + (\alpha_1 \tau_{22})_{,2} + \alpha_{2,1} \tau_{21} - \alpha_{1,2} \tau_{11}}{\alpha_1 \alpha_2} - \frac{\tau_1 \zeta}{R_1} + \tau_{\zeta 2, \zeta} + \sigma_{\zeta 2} - \left(1 + \frac{\zeta}{R_2}\right) \sigma_{2 \zeta} = 0, \quad (3e)$$

$$\frac{(\alpha_2 \tau_{1 \zeta})_{,1} + (\alpha_1 \tau_{2 \zeta})_{,2}}{\alpha_1 \alpha_2} - \frac{\tau_{21}}{R_2} + \frac{\tau_{12}}{R_1} + \tau_{\zeta \zeta, \zeta} + \left(1 + \frac{\zeta}{R_1}\right) \sigma_{12} - \left(1 + \frac{\zeta}{R_2}\right) \sigma_{21} = 0. \quad (3f)$$

We complement the differential equations (3) by the statement of boundary conditions for the faces  $\zeta = \pm c$  of the layer in the following form

$$\sigma_{\zeta i}(\pm c) = \pm \frac{p_i}{2} + \frac{q_i^\sigma}{2c}, \quad \sigma_{\zeta \zeta}(\pm c) = \pm \frac{p_\zeta}{2} + \frac{q_\zeta^\sigma}{2c}, \quad (4a, b)$$

$$\tau_{\zeta i}(\pm c) = \pm \frac{q_i^\tau}{2c}, \quad \tau_{\zeta \zeta}(\pm c) = \pm \frac{q_\zeta^\tau}{2c}. \quad (4c, d)$$

We note that insofar as the two-dimensional theory is concerned, which is the object of this analysis, the traction contributions  $p_i$  and  $p_\zeta$  will come out to be the ordinary force load intensity components, while the traction contributions  $q_i^\sigma$  and  $q_\zeta^\tau$  will come out to be ordinary moment load intensity components turning about the tangent vectors to the shell surface coordinate curves. At the same time, the traction component  $q_\zeta^\tau$  will be a moment load intensity component turning about the normal to the shell surface, and the traction component  $q_\zeta^\sigma$  will be what might be termed a "thickness-changing" kind of force load intensity component.



### The Two-Dimensional Equilibrium Equations

We introduce force-stress resultants  $N_{ij}$  and  $Q_i$  in the usual form

$$N_{ij} = \int_{-c}^c \sigma_{ij} d\zeta, \quad Q_i = \int_{-c}^c \sigma_{i\zeta} d\zeta. \quad (5a, b)$$

Simple integration with respect to  $\zeta$  of the force stress differential equations (3a, b, c), and observation of the boundary conditions (4a, b), then gives the conventional two-dimensional equations of force equilibrium

$$\frac{(\alpha_2 N_{11})_{,1} + \dots}{\alpha_1 \alpha_2} + \frac{Q_1}{R_1} + p_1 = 0, \quad \frac{(\alpha_2 N_{12})_{,1} + \dots}{\alpha_1 \alpha_2} + \frac{Q_2}{R_2} + p_2 = 0, \quad (6a, b)$$

$$\frac{(\alpha_2 Q_1)_{,1} + (\alpha_1 Q_2)_{,2}}{\alpha_1 \alpha_2} - \frac{N_{11}}{R_1} - \frac{N_{22}}{R_2} + p_\zeta = 0. \quad (6c)$$

We next introduce force-stress couples  $M_{ij}^\sigma$  and moment-stress resultants  $M_{ij}^\tau$  and  $P_i$  defined by

$$M_{ij}^\sigma = \int_{-c}^c \sigma_{ij} \zeta d\zeta, \quad M_{ij}^\tau = \int_{-c}^c \tau_{ij} d\zeta, \quad P_i = \int_{-c}^c \tau_{i\zeta} d\zeta. \quad (5c, d, e)$$

In attempting to obtain equilibrium differential equations for the  $M_{ij}^\sigma$ ,  $M_{ij}^\tau$  and  $P_i$  through appropriate integration of the three dimensional equilibrium equations (3a) to (3f) we find it necessary to introduce supplementary two-dimensional force-stress measures  $Q_i^*$ ,  $S_i$ ,  $T$ , of the form

$$Q_i^* = \int_{-c}^c (\sigma_{\zeta i} - \frac{\zeta}{R_i} \sigma_{i\zeta}) d\zeta, \quad S_i = \int_{-c}^c \sigma_{i\zeta} \zeta d\zeta, \quad T = \int_{-c}^c \sigma_{\zeta\zeta} d\zeta. \quad (5f, g, h)$$

With this we obtain from equations (3a) to (3c), with  $\int_{-c}^c \zeta \sigma_{\zeta i} d\zeta$   
 $= \zeta \sigma_{\zeta i} \Big|_{-c}^c - \int_{-c}^c \sigma_{\zeta i} d\zeta$ , and with a corresponding relation involving  $\sigma_{\zeta \zeta}$ ,

$$\frac{(\alpha_2 M_{11}^\sigma)_{,1} + (\alpha_1 M_{21}^\sigma)_{,2} + \alpha_{1,2} M_{12}^\sigma - \alpha_{2,1} M_{22}^\sigma}{\alpha_1 \alpha_2} - Q_1^* + q_1^\sigma = 0, \quad (6d)$$

$$\frac{(\alpha_2 M_{12}^\sigma)_{,1} + (\alpha_1 M_{22}^\sigma)_{,2} + \alpha_{2,1} M_{21}^\sigma - \alpha_{1,2} M_{11}^\sigma}{\alpha_1 \alpha_2} - Q_2^* + q_2^\sigma = 0, \quad (6e)$$

$$\frac{(\alpha_2 S_1)_{,1} + (\alpha_1 S_2)_{,2}}{\alpha_1 \alpha_2} - \frac{M_{11}^\sigma}{R_1} - \frac{M_{22}^\sigma}{R_2} - T + q_\zeta^\sigma = 0. \quad (6f)$$

At the same time we obtain from the three-dimensional moment equilibrium equations (3d, e, f),

$$\frac{(\alpha_2 M_{11}^\tau)_{,1} + (\alpha_1 M_{21}^\tau)_{,2} + \alpha_{1,2} M_{12}^\tau - \alpha_{2,1} M_{22}^\tau}{\alpha_1 \alpha_2} + \frac{P_2}{R_2} + Q_1^* - Q_1 + q_1^\tau = 0, \quad (6g)$$

$$\frac{(\alpha_2 M_{12}^\tau)_{,1} + (\alpha_1 M_{22}^\tau)_{,2} + \alpha_{2,1} M_{21}^\tau - \alpha_{1,2} M_{11}^\tau}{\alpha_1 \alpha_2} - \frac{P_1}{R_1} + Q_2^* - Q_2 + q_2^\tau = 0, \quad (6h)$$

$$\frac{(\alpha_2 P_1)_{,1} + (\alpha_1 P_2)_{,2}}{\alpha_1 \alpha_2} + \frac{M_{12}^\tau}{R_1} - \frac{M_{21}^\tau}{R_2} + N_{12} - N_{21} + \frac{M_{12}^\sigma}{R_1} - \frac{M_{21}^\sigma}{R_2} + q_\zeta^\tau = 0, \quad (6i)$$

giving us altogether nine two-dimensional differential equations of equilibrium.

The following observations may be made concerning the above system of nine equilibrium equations.

(1) We recover the customary six equations of force and moment equilibrium for the case of absent moment stresses, that is for the case that  $M_{ij}^T = P_i = 0$  and  $q_i^T = q_\zeta^T = 0$ , through use of all nine of the above equations. Three of these, equations (6a, b, c) remain as they are. Two of them, (6d, e), assume the conventional form upon setting  $M_{ij}^\sigma = M_{ij}$  and upon using the moment stress resultant equations (6g, h) for the purpose of eliminating the quantities  $Q_i^*$  from (6d, e). Finally, the conventional sixth equation, expressing moment equilibrium about the normals to the shell surface, follows here directly as the third moment stress resultant equation, without supplementary identity considerations as in the conventional derivations of this equation.

(2) Even in the absence of moment stresses there remains a "seventh" equation, involving the thickness-changing force load intensity measure  $q_\zeta^\sigma$ , as well as the three transverse force-stress measures  $S_i$  and  $T$ . We note that this seventh equation has previously been shown to be of some significance for the analysis of sandwich-type shells [2, 5].

#### Strain Displacement Relations

We introduce displacement measures in a way to enable all nine surface load intensity measures to do work and, consistent with this, we introduce strain measures in a way to enable all twenty-one stress measures to do work. Assuming for simplicity's sake at this point the vanishing of all displacements along the edges of the shell we then have a virtual work relation



$$\begin{aligned}
& \int (p_i \delta u_i + p_\zeta \delta w + q_i^\sigma \delta \phi_i + q_i^\tau \delta \psi_i + q_\zeta^\sigma \delta \rho + q_\zeta^\tau \delta w) dA \\
& = \int \{ N_{ij} \delta \epsilon_{ij} + Q_i \delta \gamma_i + Q_i^* \delta \gamma_i^* + M_{ij}^\sigma \delta \chi_{ij}^\sigma + M_{ij}^\tau \delta \chi_{ij}^\tau \\
& \quad + S_i \delta \mu_i + P_i \delta \lambda_i + T \delta \epsilon_\zeta \} dA, \tag{7}
\end{aligned}$$

with  $dA = \alpha_1 \alpha_2 d\xi_1 d\xi_2$ , which is to hold subject to the nine equilibrium equations (6). Elimination of the quantities  $p$  and  $q$  in (7), through use of equations (6), and suitable integrations by parts, in order to eliminate derivatives of the twenty-one measures of stress, with these measures now being arbitrary, then leads to twenty-one virtual strain displacement relations which, because of linearity, may be translated immediately into actual strain displacement relations, as follows

$$\epsilon_{11} = \frac{u_{1,1}}{\alpha_1} + \frac{\alpha_{1,2} u_2}{\alpha_1 \alpha_2} + \frac{w}{R_1}, \quad \epsilon_{12} = \frac{u_{2,1}}{\alpha_1} - \frac{\alpha_{1,2} u_1}{\alpha_1 \alpha_2} - w, \text{ etc.} \tag{8a-d}$$

$$\gamma_i = \frac{w_{,i}}{\alpha_i} - \frac{u_i}{R_i} + \psi_i, \quad \gamma_i^* = \phi_i - \psi_i, \tag{9a, b}$$

$$\chi_{11}^\sigma = \frac{\phi_{1,1}}{\alpha_1} + \frac{\alpha_{1,2} \phi_2}{\alpha_1 \alpha_2} + \frac{\rho}{R_1}, \quad \chi_{12}^\sigma = \frac{\phi_{2,1}}{\alpha_1} - \frac{\alpha_{1,2} \phi_1}{\alpha_1 \alpha_2} - \frac{w}{R_1}, \text{ etc.} \tag{10a-d}$$

$$\chi_{11}^\tau = \frac{\psi_{1,1}}{\alpha_1} + \frac{\alpha_{1,2} \psi_2}{\alpha_1 \alpha_2}, \quad \chi_{12}^\tau = \frac{\psi_{2,1}}{\alpha_1} - \frac{\alpha_{1,2} \psi_1}{\alpha_1 \alpha_2} - \frac{w}{R_1}, \text{ etc.} \tag{11a-d}$$

$$\epsilon_\zeta = \rho, \quad \mu_i = \frac{\rho_{,i}}{\alpha_i}, \quad \lambda_1 = \frac{w_{,1}}{\alpha_1} + \frac{\psi_2}{R_1}, \quad \lambda_2 = \frac{w_{,2}}{\alpha_2} - \frac{\psi_1}{R_2}. \tag{12a-e}$$



We note that these strain displacement relations imply distinct rotational displacement measures, corresponding to the moments  $M_{ij}^{\sigma}$  and  $M_{ij}^{\tau}$ , respectively, except for the case for which constraint conditions of the form  $\gamma_i^* = 0$  may be assumed. Furthermore, we have the occurrence of the thickness-changing displacement measure  $\rho$ , being an effective transverse normal strain measure, with derivatives which are a measure of what might be called anti-transverse shearing strain.

The above formulation of two-dimensional theory including the effect of moment stresses is more general than the theory obtained earlier by asymptotic considerations of three-dimensional theory in conjunction with certain order-of-magnitude stipulations concerning moment stress and force stress constitutive coefficients [4]. A "direct" two-dimensional theory corresponding to the one considered in [4] may be deduced within the present context by introducing the combined moments  $M_{ij} = M_{ij}^{\sigma} + M_{ij}^{\tau}$  and the combined loads  $q_i^{\sigma} + q_i^{\tau}$  and by replacing the four moment equilibrium equations (6d, e, g, h) by the two combined equations

$$\frac{(\alpha_2 M_{11})_{,1} + \dots}{\alpha_1 \alpha_2} - Q_1 + q_1 = 0, \quad \frac{(\alpha_2 M_{12})_{,1} + \dots}{\alpha_1 \alpha_2} - Q_2 + q_2 = 0, \quad (13a, b)$$

leaving, altogether, a system of seven equilibrium equations and leading, with  $\gamma_i^* = 0$  and  $\kappa_{ij}^{\sigma} = \kappa_{ij}^{\tau} = \kappa_{ij}$ , to a system of fifteen strain displacement relations, instead of twenty-one, involving the displacement measures  $u_i$ ,  $w$ ,  $\phi_i$ ,  $\omega$  and  $\rho$ . Evidently, this direct theory is meaningful only to the extent that it is justified by the three-dimensional considerations in [4].

An extreme case of the order-of-magnitude restrictions on moment stress constitutive coefficients in [4] is given when it is assumed that the three-dimensional medium is unable to support any moment stresses and moment loads. We then have  $M_{ij}^\tau = P_i = q_i^\tau = q_\zeta^\tau = 0$  and the nine equilibrium equations (6), with  $M_{ij}^\sigma = M_{ij}$ ,  $Q_i^* = Q_i$ , and  $q_i^\sigma = q_i$  altogether become a system of seven equations. It is noteworthy that one of these; the "third" moment equilibrium equation (6i), remains a consequence of the three-dimensional moment stress equilibrium equations, in spite of the fact that moment stresses are assumed absent at the outset. Aside from this, the significant content of our present derivation for this class of cases is the appearance of the seventh, thickness-changing, equilibrium equation (6f), involving the transverse normal stress measure  $T$  and the anti-transverse shear stress measures  $S_i$ .

The strain displacement relations for this theory, again derived through use of the principle of virtual work (with the "non-existent" twisting moment load component  $q_\zeta^\tau$  retained for the duration of this derivation) now consist of the four relations (8) for the components  $\epsilon_{ij}$ , together with the two relations  $v_i = w_{,i}/\alpha_i - u_i/R_i + \phi_i$ , with these six relations being of the conventional form [3], and of the seven additional relations

$$\kappa_{11} = \frac{\phi_{1,1}}{\alpha_1} + \frac{\alpha_{1,2}\phi_2}{\alpha_1\alpha_2} + \frac{\rho}{R_1}, \quad \kappa_{12} = \frac{\phi_{2,1}}{\alpha_1} - \frac{\alpha_{1,2}\phi_1}{\alpha_1\alpha_2} - \frac{w}{R_1}, \text{ etc.} \quad (14a-d)$$

$$\epsilon_\zeta = \rho, \quad \mu_1 = \frac{\rho_{,1}}{\alpha_1}, \quad \mu_2 = \frac{\rho_{,2}}{\alpha_2}, \quad (12a-c)$$

which are distinguished from the conventional relations by the appearance of the displacement measure  $\rho$  and of the three strain measures  $\epsilon_\zeta$  and  $\mu_i$ .

### Stress Strain Relations

A special case of the theory without moment stresses, as discussed at the end of the preceding section, is given by the case of a sandwich-type shell with stress strain relations of the form [5]

$$\epsilon_{11} = C(N_{11} - \nu N_{22}), \quad \epsilon_{21} = (1 + \nu)CN_{12}, \quad \epsilon_{12} = \dots, \quad \epsilon_{22} = \dots, \quad (13)$$

$$M_{11} = D(\kappa_{11} + \nu \kappa_{22}), \quad M_{12} = M_{21} = (1 - \nu)D(\kappa_{12} + \kappa_{21}), \quad M_{22} = \dots,$$

$$\gamma_i = C_Q Q_i, \quad \epsilon_\zeta = C_T T, \quad S_i = 0, \quad (14)$$

with the  $\epsilon_{ij}$  as in (8a-d), the  $\kappa_{ij}$  as in (14a-d),  $\epsilon_\zeta$  as in (12a) and the  $\gamma_i$  as in (9a, b) with  $\gamma_i^* = 0$ . We note once more that the significant "non-conventional" aspect of these equations of two-dimensional shell theory is the appearance of the displacement variable  $\rho$  in the expressions for  $\kappa_{11}$ ,  $\kappa_{22}$  and  $\epsilon_\zeta$ , in conjunction with the seventh equilibrium equation (6f), for  $M_{11}$ ,  $M_{22}$  and  $T$ .

We consider as a second special case the problem of in-plane deformations of flat plates in terms of cartesian-coordinate independent variables, as governed by the conventional equilibrium equations

$$N_{11,1} + N_{21,2} + p_1 = 0, \quad N_{12,1} + N_{22,2} + p_2 = 0, \quad (15a, b)$$

where  $N_{12} = N_{21}$ , together with the unconventional equilibrium equation

$$S_{1,1} + S_{2,2} - T + q_\zeta = 0, \quad (15c)$$



and together with stress strain relations of the form

$$N_{ij} = B_{ij}^{mn} \epsilon_{mn} + B_{ij}^{\zeta} \epsilon_{\zeta}, \quad T = B_{ij}^{\zeta} \epsilon_{ij} + B_{\rho}^{\zeta} \epsilon_{\zeta}, \quad S_i = B_{ij} \mu_j \quad (16)$$

with  $B_{ij}^{mn} = B_{ji}^{mn} = B_{ij}^{nm} = B_{ji}^{nm}$ ,  $B_{ij}^{\zeta} = B_{ji}^{\zeta}$ ,  $B_{ij} = B_{ji}$ , and with  $\epsilon_{11} = u_{1,1}$ ,  
 $\epsilon_{12} + \epsilon_{21} = u_{2,1} + u_{1,2}$ ,  $\epsilon_{22} = u_{2,2}$ ,  $\epsilon_{\zeta} = \rho$ ,  $\mu_j = \rho_{,j}$ .

It is evident that the three equilibrium equations (15) in conjunction with the stress strain relations (16) represent a generalization of the conventional fourth order plane stress problem for the two displacement components  $u_i$ , to a sixth order problem for the three displacement components  $u_i$  and  $\rho$ . What is not evident at this point is whether there are in fact three-dimensional problems for plane elastic layers, of such nature that the conventional two-dimensional plane stress approximation in terms of the  $N_{ij}$  is not adequate, while at the same time the two-dimensional problem with consideration of the supplementary stress measures  $S_i$  and  $T$  is in fact adequate. The Section which follows is intended as a starting point towards an answer to this question.

#### In-Plane Stretching of Symmetrically Laminated Sheet

We consider a plane layer with three-dimensional stress strain relations

$$\begin{aligned} \sigma_{11} &= E_{11} e_{11} + E_{12} e_{22} + E_{13} \gamma_{12} + E_{1\zeta} e_{\zeta\zeta}, & \sigma_{22} &= E_{21} e_{11} + \dots \\ \sigma_{12} &= E_{31} e_{11} + \dots + E_{3\zeta} e_{\zeta\zeta}, & \sigma_{\zeta\zeta} &= E_{1\zeta} e_{11} + \dots + E_{3\zeta} \gamma_{12} + E_{\zeta\zeta} e_{\zeta\zeta} \end{aligned} \quad (17)$$

where  $e_{11} = U_{1,1}$ ,  $e_{22} = U_{2,2}$ ,  $\gamma_{12} = U_{1,2} + U_{2,1}$ ,  $e_{\zeta\zeta} = W_{,\zeta}$ , and



$$\sigma_1 \zeta = G_{11} \gamma_{1\zeta} + G_{12} \gamma_{2\zeta}, \quad \sigma_2 \zeta = G_{12} \gamma_{1\zeta} + G_{22} \gamma_{2\zeta} \quad (18)$$

where  $\gamma_{i\zeta} = U_{i,\zeta} + W_{,i}$ , where the  $E$  and  $G$  are given even functions of the thickness coordinate  $\zeta$ , and where we wish to establish two-dimensional stress strain relations involving the resultants  $N_{ij}$ ,  $S_i$  and  $T$ .

An inspection of the given three-dimensional relations indicates that the essential difficulty now consists in evaluating the associated strain integrals. We may overcome this difficulty by approximating the three-dimensional displacement functions  $U_i$  and  $W$  in the form

$$U_i = u_i(x_1, x_2), \quad W = (\zeta/c)w(x_1, x_2) \quad (19a, b)$$

Therewith, we obtain from equations (17),

$$\begin{aligned} N_{11} &= u_{1,1} \int_{-c}^c E_{11} d\zeta + \dots + (u_{1,2} + u_{2,1}) \int_{-c}^c E_{13} d\zeta + \frac{w}{c} \int_{-c}^c E_1 \zeta d\zeta \\ N_{22} &= \dots, \quad N_{12} = \dots, \quad T = u_{1,1} \int_{-c}^c E_1 \zeta d\zeta + \dots + \frac{w}{c} \int_{-c}^c E \zeta \zeta d\zeta \end{aligned} \quad (20)$$

and from equations (18)

$$S_1 = \frac{w_{,1}}{c} \int_{-c}^c \zeta^2 G_{11} d\zeta + \frac{w_{,2}}{c} \int_{-c}^c \zeta^2 G_{12} d\zeta, \quad S_2 = \dots \quad (21)$$

and it is evident that we may identify equations (20) and (21) with equations (16) upon identifying the displacement variable  $\rho$  in (16) with the quantity  $w/c$  in (20) and (21).

Having equations (20) and (21) in conjunction with the three equilibrium equations (15a, b, c) we may recover the conventional theory by assuming that  $T$  is negligibly small in the stress strain relations (20) and use this assumption to eliminate  $w/c$  from the remaining contents of (20), with the conventional problem of determining the  $N_{ij}$  and  $u_i$  remaining. Subsequently we may use (21) to obtain the  $S_i$ , and then (15c) in order to determine a consistent approximation for  $T$ .

In the event that a quantitative improvement of the above approximation is desired the question arises whether the system (15) in conjunction with (20) and (21) is in fact such an improvement or whether the explicit consideration of the  $W$ -approximation (19b) in the shear relations (21) does not in effect require the simultaneous consideration of a refined  $U_i$ -approximation, of the form

$$U_i = u_i(x_1, x_2) + (\zeta/c)^2 v_i(x_1, x_2). \quad (19a')$$

Use of (19a') in place of (19a) changes equations (21) into

$$S_i = \left( \frac{w}{c} + 2 \frac{v}{c^2} \right) \int_{-c}^c \zeta^2 G_{11} d\zeta + \left( \frac{w}{c} + 2 \frac{v}{c^2} \right) \int_{-c}^c \zeta^2 G_{12} d\zeta, \text{ etc} \quad (21')$$

and equations (20) become

$$\begin{aligned} N_{11} &= u_{1,1} \int_{-c}^c E_{11} d\zeta + \frac{v_{1,1}}{c^2} \int_{-c}^c \zeta^2 E_{11} d\zeta + \dots + \frac{w}{c} \int_{-c}^c E_1 \zeta d\zeta \\ N_{22} &= \dots, \dots; T = u_{1,1} \int_{-c}^c E_1 \zeta d\zeta + \frac{v_{1,1}}{c^2} \int_{-c}^c \zeta^2 E_1 \zeta d\zeta + \dots \end{aligned} \quad (20')$$

Evidently, the appearance of the additional displacement variables  $v_i$  in the stress strain relations of the problem means that the three equilibrium equations (15) are now inadequate. We may obtain supplementary equations by considering supplementary weighted averages of the given three-dimensional equilibrium equations. In doing this we find that it will now not be possible to incorporate the resulting consequences into our initial scheme involving use of the principle of virtual work for the derivation of strain displacement relations for a Cosserat-type surface theory. We bypass this difficulty by introducing supplementary stress resultants  $R_{ij}$  in the form

$$R_{ij} = \int_{-c}^c (1 - \zeta^2/c^2) \sigma_{ij} d\zeta, \quad (22)$$

and we obtain two equilibrium equations involving the  $R_{ij}$ , in conjunction with the  $S_i$ , from the relations  $\int_{-c}^c (\sigma_{ij,i} + \sigma_{\zeta j, \zeta})(1 - \zeta^2/c^2) d\zeta = 0$  in the form

$$R_{ij,i} + (2/c^2) S_j = 0 \quad (23)$$

Note that in doing this we did not need to introduce any supplementary measures of transverse shearing stress.

Having equations (23) we now complete our system of equations of two-dimensional theory by deducing, from equations (17), the supplementary stress strain relations

$$R_{11} = u_{1,1} \int_{-c}^c (1 - \frac{\zeta^2}{c^2}) E_{11} d\zeta + v_{1,1} \int_{-c}^c (\frac{\zeta^2}{c^2} - \frac{\zeta^4}{c^4}) E_{11} d\zeta + \dots$$



$$+ \frac{w}{c} \int_{-c}^c (1 - \frac{\zeta^2}{c^2}) E_1 d\zeta, \quad R_{12} = \dots, \quad R_{22} = \dots, \quad (24)$$

where we note specifically that only the first three of the four relations in (17) are made use of in this fashion.

We refrain from extending the above in various possible ways, such as to the case for which transverse bending and stretching are coupled because of material asymmetry, and such as the use of more sophisticated averaging schemes, consistent with what would follow from a consideration of the laminated sheet problem with the help of direct (Rayleigh-Ritz) methods on the basis of the principle of minimum complementary energy. Instead we recall once more the principal purpose of this note, to help bridge the gap between "ordinary" and "Cosserat-type" two-dimensional theories, with an indication of apparent limitations on the applicability of results of the Cosserat type.

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